

Gauge-Invariant Variables on Cosmological Hypersurfaces

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We show how gauge-invariant cosmological perturbations may be constructed by an unambiguous choice of hypersurface-orthogonal time-like vector field (i.e., time-slicing). This may be defined either in terms of the metric quantities such as curvature or shear, or using some matter field. As an example, we show how linear perturbations in the covariant fluid-flow approach can then be presented in an explicitly gauge-invariant form in the coordinate based formalism.

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In an unperturbed Friedmann-Robertson-Walker (FRW) universe the homogeneous spatial hypersurfaces pick out a natural cosmic time coordinate, and hence a (3+1) decomposition of spacetime. But in the presence of inhomogeneities this choice of coordinates is no longer unambiguous.

The need to clarify these ambiguities lead to the fluid-flow approach which uses the velocity field of the matter to define perturbed quantities orthogonal to the fluid-flow [1–3]. An alternative school has sought to define gauge-invariant perturbations in any coordinate system by constructing quantities that are explicitly invariant under general coordinate transformations [4–6]. Results obtained in the two formalisms can be difficult to compare. One stresses the virtue of invariance of metric perturbations under gauge transformations, while the other claims to be covariant and therefore manifestly gauge-invariant due to its physically transparent definition.

The purpose of this brief report is to stress that linear perturbations defined on an unambiguous physical choice of hypersurface can always be written in a gauge-invariant manner. In the coordinate based formalism [4] the choice of hypersurface implies a particular choice of gauge, but by including the explicit gauge transformation from an arbitrary initial coordinate system the metric perturbations can be given in an explicitly gauge-invariant form. Similar conclusions were presented in a recent paper by Unruh [7] while the present paper was in preparation. In this language, linear perturbations in the fluid-flow or “covariant” approach appear as a particular gauge choice (the comoving orthogonal gauge [4,5]) whose metric perturbations can be given in an explicitly gauge-invariant form. But there are many other possible choices of hypersurface, including the zero-shear orthogonal (or longitudinal) gauge [4–6,8] in which gauge-invariant quantities may be defined.

I. THE METRIC APPROACH

We will restrict our analysis to the case where the perturbations can be constructed from scalar quantities defined on 3-D hypersurfaces [4,9]. The line element al-

lowing arbitrary linear scalar perturbations of a FRW background can be written

$$ds^2 = a^2(\eta) \left\{ -(1 + 2\phi)d\eta^2 + 2B_{|i}d\eta dx^i + [(1 - 2\psi)\gamma_{ij} + 2E_{|ij}]dx^i dx^j \right\}, \quad (1.1)$$

where we use the conformal time η . The spatial metric on 3-spaces of constant curvature is given by γ_{ij} and covariant derivatives with respect to this metric are denoted by $X_{|ij}$. *

The intrinsic spatial curvature on hypersurfaces of constant conformal time η is given by [4,5]

$${}^{(3)}R = \frac{6\kappa}{a^2} + \frac{12\kappa}{a^2}\psi + \frac{4}{a^2}\psi_{|i}{}^i \quad (1.3)$$

For a perturbation with comoving wavenumber k we therefore have

$$\delta^{(3)}R = \frac{4}{a^2} (3\kappa - k^2) \psi \quad (1.4)$$

and ψ is often simply referred to as the curvature perturbation.

The metric perturbations ϕ , ψ , E and B can also be related to various geometrical quantities defined in terms of the unit time-like vector field

$$N^\mu = \frac{1}{a}(1 - \phi, B_{|i}). \quad (1.5)$$

*Our notation coincides with that of Mukhanov, Feldman and Brandenberger [6] which is widely used in the literature. For comparison with the notation of Bardeen [4] note that

$$\begin{aligned} \phi &\equiv A Q^{(0)}, & \psi &\equiv -\left(H_L + \frac{1}{3}H_T\right) Q^{(0)}, \\ B &\equiv \frac{B_B Q^{(0)}}{k}, & E &\equiv \frac{H_T Q^{(0)}}{k^2}, \end{aligned} \quad (1.2)$$

where Bardeen explicitly included $Q^{(0)}(x^i)$, the eigenmodes of the spatial Laplacian with eigenvalue $-k^2$.

which is orthogonal to the constant- η hypersurfaces. We can write the expansion, acceleration and shear of the vector field [5]

$$\theta = 3\frac{a'}{a^2}(1-\phi) - \frac{3}{a}\psi' - \frac{1}{a}(B-E')_{|i}^i, \quad (1.6)$$

$$a_i = \phi_{|i}, \quad (1.7)$$

$$\sigma_{ij} = a\left(\sigma_{|ij} - \frac{1}{3}\gamma_{ij}\sigma_{|k}^k\right), \quad (1.8)$$

where the scalar describing the shear is

$$\sigma = -B + E'. \quad (1.9)$$

Note that all these physical quantities can be written in terms of just three scalars ϕ , ψ and σ .

The homogeneity of a FRW spacetime gives a natural choice of coordinates in the absence of perturbations. But in the presence of linear perturbations we are free to make a first-order change in the coordinates, i.e., a gauge transformation,

$$\tilde{\eta} = \eta + \xi^0, \quad \tilde{x}^i = x^i + \xi_{|}^i, \quad (1.10)$$

where ξ and ξ^0 are arbitrary scalar functions. A scalar transformation of this form preserves the scalar nature of the metric perturbations [4,9]. The function ξ^0 determines the choice of constant- η hypersurfaces, i.e., the time-slicing, while ξ then selects the spatial coordinates within these hypersurfaces. The choice of coordinates is arbitrary to first-order and the definitions of the first-order metric and matter perturbations are thus gauge-dependent.

The coordinate transformation of Eq. (1.10) induces a change in the functions ϕ , ψ , B and E defined by Eq. (1.1)

$$\tilde{\phi} = \phi - h\xi^0 - \xi^{0'} \quad (1.11)$$

$$\tilde{\psi} = \psi + h\xi^0 \quad (1.12)$$

$$\tilde{B} = B + \xi^0 - \xi' \quad (1.13)$$

$$\tilde{E} = E - \xi \quad (1.14)$$

where $h = a'/a$ and a dash indicates differentiation with respect to conformal time η . Any scalar φ (including the fluid density or pressure) which is homogeneous in the background FRW model can be written as $\varphi(\eta, x^i) = \varphi_0(\eta) + \delta\varphi(\eta, x^i)$. The perturbation then transforms as

$$\delta\tilde{\varphi} = \delta\varphi - \xi^0\varphi'_0, \quad (1.15)$$

Physical scalars on the hypersurfaces, such as the curvature, acceleration, shear or $\delta\varphi$, only depend on the choice of ξ^0 , but are independent of the coordinates within the 3-D hypersurfaces determined by ξ . The function ξ can only affect the components of 3-vectors or 3-tensors on the hypersurfaces and not 3-scalars.

The gauge-dependence of the metric perturbations lead Bardeen to propose that only quantities that are explicitly gauge-invariant under gauge transformations should

be considered. The two scalar gauge functions allow two of the metric perturbations to be eliminated implying that one should seek two remaining gauge-invariant combinations. By studying the transformation Eqs. (1.11–1.14), Bardeen constructed two such quantities [4,6][†]

$$\Phi \equiv \phi + h(B - E') + (B - E')', \quad (1.16)$$

$$\Psi \equiv \psi - h(B - E'). \quad (1.17)$$

These turn out to coincide with the metric perturbations in a particular gauge, called variously the orthogonal zero-shear [4,5], conformal Newtonian [8] or longitudinal gauge [6]. It may therefore appear that this gauge is somehow preferred over other choices. However any unambiguous choice of time-slicing can be used to define explicitly gauge-invariant perturbations. The longitudinal gauge of Ref. [6] provides but one example.

If we choose to work on spatial hypersurfaces with vanishing shear $\tilde{\sigma}$ this implies that starting from arbitrary coordinates we should perform a gauge-transformation

$$\xi_l^0 = -B + E'. \quad (1.18)$$

This is sufficient to determine the ϕ , ψ , σ or any other scalar quantity on these hypersurfaces. In addition, the longitudinal gauge is completely determined by the spatial gauge choice

$$\xi_l = E, \quad (1.19)$$

and hence $\tilde{E} = \tilde{B} = 0$. The remaining functions ϕ , ψ and $\delta\varphi$ become

$$\tilde{\phi}_l = \phi + h(B - E') + (B - E')', \quad (1.20)$$

$$\tilde{\psi}_l = \psi - h(B - E'), \quad (1.21)$$

$$\delta\tilde{\varphi}_l = \delta\varphi + \varphi'_0(B - E'). \quad (1.22)$$

Note, that $\tilde{\phi}_l$ and $\tilde{\psi}_l$ are then identical to Φ and Ψ defined in Eqs. (1.16) and (1.17). These gauge-invariant quantities are simply a coordinate independent definition of the perturbations in the longitudinal gauge. *Other specific gauge choices may equally be used to construct quantities that are manifestly gauge-invariant.*

An interesting alternative gauge choice, defined purely by local metric quantities is the uniform curvature gauge [5,10,11], also called the off-diagonal gauge [12]. In this gauge one selects spatial hypersurfaces on which the induced 3-metric is left unperturbed, which requires $\tilde{\psi} = \tilde{E} = 0$. This corresponds to a gauge transformation

$$\xi_\kappa^0 = -\frac{\psi}{h}, \quad \xi_\kappa = E. \quad (1.23)$$

The gauge-invariant definitions of the remaining metric degrees of freedom are then from Eqs. (1.11) and (1.13)

[†]In Bardeen's notation these gauge-invariant perturbations are given as $\Phi \equiv \Phi_A Q^{(0)}$ and $\Psi \equiv -\Phi_H Q^{(0)}$.

$$\tilde{\phi}_\kappa = \phi + \psi + \left(\frac{\psi}{h}\right)', \quad (1.24)$$

$$\tilde{B}_\kappa = B - E' - \frac{\psi}{h}, \quad (1.25)$$

These gauge-invariant combinations were denoted \mathcal{A} and \mathcal{B} by Kodama and Sasaki [5]. Perturbations of scalar quantities in this gauge become

$$\delta\tilde{\varphi}_\kappa = \delta\varphi + \varphi'_0 \frac{\psi}{h}. \quad (1.26)$$

In some circumstances it is actually more convenient to use these alternative gauge-invariant variables instead of Φ and Ψ . For instance, when calculating the evolution of perturbations during a collapsing “pre Big Bang” era the perturbations $\tilde{\phi}_\kappa$ and \tilde{B}_κ may remain small even when Φ and Ψ become large [12]. Note that Eq. (1.26) gives the gauge-invariant scalar field perturbation introduced by Mukhanov [13].

For comparison note that the synchronous gauge, defined by $\phi = \tilde{B} = 0$, does not determine the time-slicing unambiguously. There is a residual gauge freedom $\xi^0 = X/a$, where $X(x^i)$ is an arbitrary function of the spatial coordinates, and it is not possible to define gauge-invariant quantities in general using this gauge condition [14].

II. THE FLUID-FLOW APPROACH

Thus far we have concerned ourselves solely with the metric and its representation under different choices of coordinates. However in any non-vacuum spacetime we will also have matter fields to consider. Like the metric, the coordinate representation of these fields will also be gauge-dependent.

The stress-energy tensor of a perfect fluid with density ϵ , isotropic pressure p and 4-velocity u^μ is given by

$$T^\mu_\nu = (\epsilon + p) u^\mu u_\nu + p \delta^\mu_\nu. \quad (2.1)$$

The linearly perturbed velocity can be written as

$$u^\mu = \frac{1}{a} \left((1 - \phi), v_{|i} \right), \quad (2.2)$$

where we enforce the constraint $u_\mu u^\mu = -1$. We can introduce the velocity potential v since the flow is irrotational for scalar perturbations. We then get for the components of the stress energy tensor

$$T^0_0 = -(\epsilon_0 + \delta\epsilon) \quad (2.3)$$

$$T^0_i = (\epsilon_0 + p_0) (B + v)_{|i} \quad (2.4)$$

$$T^i_j = (p_0 + \delta p) \delta^i_j + \pi^i_j \quad (2.5)$$

where we have included the trace-free anisotropic stress tensor, $\pi^i_j = \pi_{|j}^i - \frac{1}{3} \delta^i_j \pi_{|k}^k$.

Coordinate transformations affect the split between spatial and temporal components of the matter fields and so quantities like the density, pressure and 3-velocity are gauge-dependent. Density and pressure are scalar quantities which transform as given in Eq. (1.15), but the velocity potential becomes

$$\tilde{v} = v + \xi'. \quad (2.6)$$

The anisotropic pressure, π_{ij} , is gauge-invariant.

The comoving gauge is defined by choosing spatial coordinates such that the 3-velocity of the fluid vanishes, $\tilde{v} = 0$. Orthogonality of the constant- η hypersurfaces to the 4-velocity, u^μ , then requires $\tilde{v} + \tilde{B} = 0$. From Eqs. (1.13) and (2.6) this implies

$$\begin{aligned} \xi_m^0 &= -(v + B) \\ \xi_m &= - \int v d\eta + \hat{\xi}(x^i) \end{aligned} \quad (2.7)$$

where $\hat{\xi}(x^i)$ represents a residual gauge freedom, corresponding to a constant shift of the spatial coordinates. All the physical quantities like curvature, expansion, acceleration and shear are independent of $\hat{\xi}$. Applying the above transformation from arbitrary coordinates, the scalar perturbations in the comoving orthogonal gauge can be written as

$$\tilde{\phi}_m = \phi + \frac{1}{a} [(v + B) a]' \quad (2.8)$$

$$\tilde{\psi}_m = \psi - h(v + B) \quad (2.9)$$

$$\tilde{E}_m = E + \int v d\eta - \hat{\xi} \quad (2.10)$$

$$\delta\tilde{\varphi}_m = \delta\varphi - \varphi'_0 (v + B) \quad (2.11)$$

Defined in this way, these combinations are gauge-invariant under transformations of their component parts in exactly the same way as, for instance, Φ and Ψ defined in Eqs. (1.16) and (1.17), apart from the residual dependence of \tilde{E}_m upon the choice of $\hat{\xi}$.

The density perturbation on the comoving orthogonal hypersurfaces is given by Eq. (2.11) in gauge-invariant form as

$$\delta\tilde{\epsilon}_m = \delta\epsilon - \epsilon'_0 (v + B), \quad (2.12)$$

and corresponds to the gauge-invariant density perturbation $\epsilon_m E_0 Q^{(0)}$ in the notation of Bardeen [4]. The gauge-invariant scalar density perturbation Δ introduced in Ref. [3] corresponds to $\delta\tilde{\epsilon}_{m|i} / \epsilon_0$.

If we wish to write these quantities in terms of the metric perturbations rather than the velocity potential then we can use the Einstein equations [6] to obtain

$$v + B = \frac{h\phi + \psi' - \kappa(B - E')}{h' - h^2 - \kappa}. \quad (2.13)$$

In particular we note that we can write the comoving curvature perturbation, given in Eq. (2.9), in terms of the longitudinal gauge-invariant quantities as

$$\tilde{\psi}_m = \Psi - \frac{h(h\Phi + \Psi')}{h' - h^2 - \kappa}. \quad (2.14)$$

Alternatively we could use the matter content to pick out uniform density hypersurfaces on which to define perturbed quantities. Using Eq. (1.15) we see that this implies a gauge transformation

$$\xi_\epsilon^0 = \frac{\delta\epsilon}{\epsilon_0'}, \quad (2.15)$$

and on these hypersurfaces the gauge-invariant curvature perturbation is [15,14]

$$\tilde{\psi}_\epsilon = \psi + h \frac{\delta\epsilon}{\epsilon_0'} \quad (2.16)$$

This coincides with ζ_{BST} defined in Refs. [16,14].

In Ref. [6] the gauge-invariant variable ζ_{MFB} is defined as

$$\zeta_{\text{MFB}} = \Phi + \frac{2}{3} \frac{\Phi' + \Phi h}{(1+w)h}, \quad (2.17)$$

where $w \equiv p_0/\epsilon_0$. On large scales (where we neglect spatial derivatives) and in flat-space ($\kappa = 0$) with vanishing anisotropic stresses ($\pi_j^i = 0$, which requires that $\Phi = \Psi$ [6]) all three quantities $\tilde{\psi}_m$, $\tilde{\psi}_\epsilon$ and ζ_{MFB} coincide. The curvature perturbation, in one or other of these forms, is often used to predict the amplitude of perturbations re-entering the horizon scale during the radiation or matter dominated eras in terms of perturbations that left the horizon during an inflationary epoch, because they remain constant on super-horizon scales (whose comoving wavenumber $k \ll h$) for adiabatic perturbations [17]. However, only $\tilde{\psi}_\epsilon$ (or ζ_{BST}) is constant on large scales, in the presence of anisotropic stresses, or background spatial curvature. Neglecting spatial gradients, it obeys the simple evolution equation [18]

$$\tilde{\psi}_\epsilon' = \left(\frac{p_0'}{\epsilon_0'} - \frac{\delta p}{\delta\epsilon} \right) 3h\tilde{\psi}_\epsilon. \quad (2.18)$$

The pre-factor on the right-hand-side is gauge-invariant and vanishes for adiabatic perturbations.

III. SUMMARY

It is with some trepidation that we present yet another paper attempting to clarify the gauge-dependence of cosmological perturbations. Nonetheless we feel that there is an important clarification of the coordinate based approach that has been previously overlooked, or at least left unstated. If we use the value of any physical scalar to unambiguously specify the gauge function ξ^0 , and hence the time-slicing of the perturbed spacetime, then we can write the resulting scalar metric perturbations ϕ , ψ , σ or any matter perturbation $\delta\varphi$ on this hypersurface in

a manifestly gauge-invariant way by explicitly including the transformation from an arbitrary coordinate system. If in addition we make an unambiguous choice of the spatial coordinates on these hypersurfaces, through the gauge-function ξ , then all the 3-tensor components also become gauge-invariant.

Examples of such gauge-invariant quantities can be constructed using the zero-shear (longitudinal) or comoving orthogonal gauges. One advantage of the comoving or fluid-flow approach is that the (3+1) decomposition need not be restricted to linear perturbations, and the metric perturbations in the coordinate-basis appear as a linearisation of the more general case [3]. Realising that there are other possible physical choices of hypersurfaces opens up the possibility of considering non-linear perturbations in other coordinate systems, as recently proposed by Sasaki and Tanaka [19].

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